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An improved lower bound on a positive stable block triangular preconditioner for saddle point problems☆

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ABSTRACT

In this paper, a new lower bound on a positive stable block triangular preconditioner for saddle point problems is derived; it is superior to the corresponding result obtained by Cao [Z.-H. Cao, Positive stable block triangular preconditioners for symmetric saddle point problems, Appl. Numer. Math. 57 (2007) 899–910]. A numerical example is reported to confirm the presented result.

1. Introduction

Consider the generalized saddle point problems

$$K\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite, $C \in \mathbb{R}^{m\times m}$ is symmetric positive semidefinite, and $B \in \mathbb{R}^{m\times n}$ $(m \le n)$ is of full rank. Systems of the form (1.1) arise in a variety of scientific and engineering applications, such as linear elasticity, fluid dynamics, electromagnetics, constrained quadratic programming [1–4]. We refer the reader to [5] for more applications and numerical solution techniques of (1.1).

Recently, Cao [6] discussed the following block triangular preconditioner for saddle point problems (1.1), that is,

$$\mathcal{G} = \begin{bmatrix} \hat{A} & B^T \\ 0 & \hat{C} \end{bmatrix},$$

where $\hat{A} \in \mathbb{R}^{n\times n}$ and $\hat{C} \in \mathbb{R}^{m\times m}$ are symmetric positive definite, together with a Krylov subspace iterative solver. Obviously, $\mathcal{G}$ is positive (real) stable, i.e., its eigenvalues are all real and positive. In [6], Cao has shown that the preconditioned matrix $K\mathcal{G}^{-1}$ is indefinite, with all the eigenvalues being real, and estimates have been provided for the interval containing these real eigenvalues. Numerical experiments showed that the block triangular preconditioner $\mathcal{G}$ is feasible and efficient. In this paper, we will derive a new lower bound for the positive stable block triangular preconditioner to solve saddle point problems. This result is superior to the corresponding result in [6]. Although this is a quantitative result, in some sense, it may reflect the region of the condition number of the preconditioned matrix and the actual solver convergence. It is well known that the

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spectral properties of the preconditioned matrix give important insight in the convergence behavior of the preconditioned Krylov subspace methods. In particular, for symmetric linear systems, it is desirable that the number of distinct eigenvalues, or at least the number of clusters, is small, because in this case convergence will be rapid. If there are only a few distinct eigenvalues, then optimal methods such as CG, MINRES, or GMRES will terminate (in exact arithmetic) after a small and precisely defined number of steps.

Some notation is required. For a vector x, x^T indicates its transpose. For a matrix A, $A > 0$ ($A \geq 0$) means that A is symmetric positive (semi)definite. Given two square matrices Λ_1 and Λ_2, $\mathrm{diag}(\Lambda_1, \Lambda_2)$ stands for the block diagonal matrix having Λ_1 as first block and Λ_2 as second block, and $\Lambda(A)$ denotes the set of eigenvalues of A.

2. Main results

Let

$$X^T\hat{A}^{-\frac{1}{2}}A\hat{A}^{-\frac{1}{2}}X = \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix},$$

with $\Lambda_1 < I$ and $\Lambda_2 > I$, and with $X = [X_1, X_2]$ being an orthogonal matrix. We define

$$H = \begin{bmatrix} \Lambda_1 & Q_1^T \\ Q_1 & -(\tilde{B}\tilde{B}^T + \tilde{C}) \end{bmatrix}, \tag{2.1}$$

where $Q_1 = \tilde{B}X_1(I - \Lambda_1)^{\frac{1}{2}}$, $\tilde{B} = \hat{C}^{-\frac{1}{2}}B\hat{A}^{-\frac{1}{2}}$, and $\tilde{C} = \hat{C}^{-\frac{1}{2}}C\hat{C}^{-\frac{1}{2}}$. From [6], it is not difficult to find that the bounds for the eigenvalues of the preconditioned matrix $K\mathcal{G}^{-1}$ are equivalent to the bounds for the eigenvalues of the matrix

$$\begin{bmatrix} \Lambda_2 & -Q_2^T \\ Q_2 & H \end{bmatrix},$$

where

$$Q_2 = \begin{bmatrix} 0 \\ -\tilde{B}X_2(\Lambda_2 - I)^{\frac{1}{2}} \end{bmatrix}.$$

One can refer to [6] for details.

A new lower valid bound for the distribution eigenvalue of the matrix H is obtained and is described as follows.

Proposition 2.1. *Let the order of H in (2.1) be k, let $\eta_{\min}$ be the smallest eigenvalue of Λ_1, and let $\mu_{\max}$ be the largest eigenvalue of $\tilde{B}\tilde{B}^T + \tilde{C}$. For any $u \in \mathbb{R}^k$, $u \neq 0$,*

$$(1+\theta)\eta_{\min} - \theta \leq \frac{u^THu}{u^Tu},$$

where

$$\theta = \frac{\eta_{\min} + \mu_{\max} + \sqrt{(\eta_{\min} + \mu_{\max})^2 + 4(1-\eta_{\min})\mu_{\max}}}{2(1-\eta_{\min})}.$$

Proof. Let $u = [x^T, y^T]^T$, $u \neq 0$. Then

$$\begin{aligned} u^THu &= x^T\Lambda_1 x + x^TQ_1^Ty + y^TQ_1x - y^T(\tilde{B}\tilde{B}^T + \tilde{C})y \\ &\geq x^T\Lambda_1 x - 2|x^TQ_1^Ty| - y^T(\tilde{B}\tilde{B}^T + \tilde{C})y. \end{aligned} \tag{2.2}$$

For $\forall\theta > 0$, we have

$$\begin{aligned} 2|x^TQ_1^Ty| &= 2|x^T(I-\Lambda_1)^{\frac{1}{2}}X_1^T\tilde{B}^Ty| \\ &= 2|x^T(I-\Lambda_1)^{\frac{1}{2}}X_1^T\tilde{B}^Ty| \\ &\leq \theta x^T(I-\Lambda_1)x + \frac{1}{\theta}y^T\tilde{B}\tilde{B}^Ty. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) yields

$$\begin{aligned} u^THu &\geq x^T((1+\theta)\Lambda_1 - \theta I)x - \left(\frac{1}{\theta}y^T\tilde{B}\tilde{B}^Ty + y^T(\tilde{B}\tilde{B}^T + \tilde{C})y\right) \\ &\geq ((1+\theta)\eta_{\min} - \theta)x^Tx - \frac{1}{\theta}\mu_{\max}y^Ty - \mu_{\max}y^Ty. \end{aligned} \tag{2.4}$$

Taking θ for (2.4) such that

$$(1+\theta)\eta_{\min} - \theta = -\left(\frac{1}{\theta}\mu_{\max} + \mu_{\max}\right),$$

that is,

$$(1-\eta_{\min})\theta^2 - (\eta_{\min} + \mu_{\max})\theta - \mu_{\max} = 0,$$

we obtain

$$\theta = \frac{\eta_{\min} + \mu_{\max} + \sqrt{(\eta_{\min} + \mu_{\max})^2 + 4(1-\eta_{\min})\mu_{\max}}}{2(1-\eta_{\min})} > 0.$$

With this choice, we have

$$u^T H u \geq ((1+\theta)\eta_{\min} - \theta)(x^T x + y^T y) = ((1+\theta)\eta_{\min} - \theta)u^T u. \tag{2.5}$$

This gives the lower bound. □

Proposition 2.2. *Under the conditions of Proposition 2.1,*

$$(1+\theta)\eta_{\min} - \theta \geq \min\{-2\mu_{\max}, 2\eta_{\min} - 1\}.$$

Proof. By simple computations, if $\theta \leq 1$, then

$$(1+\theta)\eta_{\min} - \theta - (2\eta_{\min} - 1) = (1-\theta)(1-\eta_{\min}) \geq 0.$$

It is easy to see that Proposition 2.2 holds.

If $\theta \geq 1$, then

$$(1+\theta)\eta_{\min} - \theta = -\left(\frac{1}{\theta}\mu_{\max} + \mu_{\max}\right) \geq -2\mu_{\max}.$$

Obviously, Proposition 2.2 also holds. □

Corollary 2.3. *Denote by $\eta_{\max}$ the largest eigenvalue of Λ_2. Then the eigenvalues μ of $K\mathcal{G}^{-1}$ satisfy*

$$(1+\theta)\eta_{\min} - \theta \leq \mu \leq \max\{\eta_{\max}, 1\},$$

where

$$\theta = \frac{\eta_{\min} + \mu_{\max} + \sqrt{(\eta_{\min} + \mu_{\max})^2 + 4(1-\eta_{\min})\mu_{\max}}}{2(1-\eta_{\min})}.$$

Proof. Let μ be an eigenvalue of $K\mathcal{G}^{-1}$ with eigenvector $\tilde{u} = [x; y]$ satisfying $\|\tilde{u}\| = 1$. By the above discussion, we have

$$\begin{cases} \Lambda_2 x - Q_2^T y = \mu x, \\ Q_2 x + H y = \mu y. \end{cases}$$

From [6], it is easy to see that the eigenvector $\tilde{u}$ is real. Hence x and y are real. Multiplying the two equations from the left-hand side by x^T and y^T, respectively, we obtain

$$\begin{cases} x^T \Lambda_2 x - x^T Q_2^T y = \mu \|x\|^2, \\ y^T Q_2 x + y^T H y = \mu \|y\|^2. \end{cases}$$

Eliminating $x^T Q_2^T y$, we obtain

$$\mu = x^T \Lambda_2 x + y^T H y.$$

The rest of the proof is similar to that of Theorem 2.3 in [6]. Hence it is omitted here. □

In [6], the following results were obtained.

Lemma 2.4 *([6]). Let the order of H in (2.1) be k. Denote by $\eta_{\min}$ the smallest eigenvalue of Λ_1 and by $\mu_{\max}$ the largest eigenvalue of $\tilde{B}\tilde{B}^T + \tilde{C}$. For any $u \in \mathbb{R}^k$, $u \neq 0$,*

$$\gamma \equiv \min\{-2\mu_{\max}, 2\eta_{\min} - 1\} \leq \frac{u^T H u}{u^T u} \leq 1.$$

Table 1
Values of n and m and the order of K.

h	n	m	Order of K
8×8	162	62	224
16×16	578	254	832
32×32	2178	1022	3200

Table 2
The region for all the eigenvalues of $K\mathcal{G}^{-1}$.

h	μ	$[\gamma, \tau]$	$[\delta, \tau]$
8×8	$[-1.1749, 1.5482]$	$[-2, 1.6893]$	$[-1.2013, 1.6893]$
16×16	$[-1.2949, 1.8920]$	$[-2, 2]$	$[-1.3131, 2]$
32×32	$[-1.4533, 2.0328]$	$[-2, 2.0896]$	$[-1.4715, 2.0896]$

Theorem 2.5 ([6]). *Denote by $\eta_{\max}$ the largest eigenvalue of Λ_2. Then the eigenvalues μ of $K\mathcal{G}^{-1}$ satisfy*

$$\gamma \equiv \min\{-2\mu_{\max}, 2\eta_{\min} - 1\} \le \mu \le \max\{\eta_{\max}, 1\}.$$

By the above discussion, it is easy to see that Lemma 2.4 [6] on the lower bound of matrix H is improved from Proposition 2.2. The associated Corollary 2.3 is superior to Theorem 2.5 [6]. Compared with Theorem 2.5 [6], Corollary 2.3 provides valid bounds for all the eigenvalues of the preconditioned matrix $K\mathcal{G}^{-1}$.

Subsequently, we consider the following example to illustrate the above result.

Example 2.6 ([7]). Consider the classic incompressible steady Stokes problem:

$$\begin{cases} -\Delta u + \operatorname{grad} p = f, & \text{in } \Omega, \\ -\operatorname{div} u = 0, & \text{in } \Omega, \end{cases} \tag{2.6}$$

with suitable boundary condition on $\partial\Omega$. The test problem is a "leaky" two-dimensional lid-driven cavity problem in a square $(0 \le x \le 1, 0 \le y \le 1)$. Using IFISS [8] to discretize (2.6), the finite element subdivision is based on uniform grids of square elements and the mixed finite element used is the bilinear constant-velocity–pressure Q_1–P_0 pair with stabilization (the stabilization parameter is $\frac{1}{4}$). The coefficient matrix generated by this package is singular because B corresponding to the discrete divergence operator is rank deficient. The nonsingular matrix K is obtained by dropping the first row of B, and the first two rows and columns of C. For the Stokes problem, the $(1, 1)$ block of the coefficient matrix corresponding to the discretization of the conservative term is symmetric positive definite. For convenience, we take three meshes h: $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. Information on the sparsity of the relevant matrices on the different mesh is given in Table 1.

In our numerical experiments, we take $\hat{A}$ as an incomplete Cholesky factorization of A:

$$A = LL^T + r, \qquad \hat{A} = LL^T,$$

with drop tolerance 0.01 [6], and $\hat{C} = C + B(LL^T)^{-1}B^T$. In this case, we find that $\eta_{\min} < 1$ and $\eta_{\max} > 1$. Let $\tau = \{\eta_{\max}, 1\}$ and $\delta = (1 + \theta)\eta_{\min} - \theta$ with

$$\theta = \frac{\eta_{\min} + \mu_{\max} + \sqrt{(\eta_{\min} + \mu_{\max})^2 + 4(1 - \eta_{\min})\mu_{\max}}}{2(1 - \eta_{\min})}.$$

By calculations, the values given in Table 2 is obtained, which are to verify the results of Corollary 2.3 and Theorem 2.5 [6]. From Table 2, it is not difficult to find that the theoretical results are in line with the results of numerical experiments. In Table 2, it is easy to see that the interval containing all the eigenvalues of $K\mathcal{G}^{-1}$ in Corollary 2.3 is more accurate than that of Theorem 2.5 [6]. That is, Theorem 2.5 [6] on the lower bound of all the eigenvalues $K\mathcal{G}^{-1}$ has been improved.

Finally, we present some iterative results to illustrate the convergence behaviors of GMRES(m). In general, there is no general rule for choosing a value of the restart parameter $m(m \ll n)$, which in practice mostly depends on a matter of experience. In our numerical experiments, for the sake of simplicity, we take $m = 20$. All tests are started from the zero vector, performed in MATLAB with machine precision 10^{-16}. The GMRES(20) iteration terminates if the relative residual error satisfies $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-6}$. In Table 3, we list the iteration numbers of GMRES(20) and preconditioned GMRES(20) applied to solve the Stokes equation with 8×8, 16×16, and 32×32 grids. The purpose of these experiments is simply to investigate the influence of the eigenvalue distribution on the convergence behavior of GMRES(20) iteration. IT denotes the number of iterations, and CPU (s) denotes the time (in seconds) required to solve a problem.

As can be seen in Table 3, all results show that the preconditioner $\mathcal{G}$ will improve the convergence of GMRES(20) efficiently. That is, the preconditioner $\mathcal{G}$ for saddle point problems (1.1) may be feasible and efficient.

Table 3
Iteration number and CPU (s) of GMRES(20).

Mesh	8 × 8		16 × 16		32 × 32	
	IT	CPU (s)	IT	CPU (s)	IT	CPU (s)
GMRES($\mathcal{G}$)	1(18)	0.0938	2(11)	0.9531	3(12)	1.5156
GMRES	63(4)	0.375	108(1)	11.7656	46(1)	5.9031

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References

[1] O. Axelsson, V.A. Barker, Finite Element Solution of Boundary Value Problems, Academic Press, Orlando, 1984.
[2] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, New York, 1991.
[3] H.C. Elman, D.J. Silvester, A.J. Wathen, Finite Elements and Fast Iterative Solvers, Oxford University Press, Oxford, 2003.
[4] W. Zulehner, Analysis of iterative methods for saddle point problems a unified approach, Math. Comp. 71 (2001) 479–505.
[5] M. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, Acta Numer. 14 (2005) 1–137.
[6] Z.-H. Cao, Positive stable block triangular preconditioners for symmetric saddle point problems, Appl. Numer. Math. 57 (2007) 899–910.
[7] S.-L. Wu, C.-X. Li, Indefinite block triangular preconditioner for symmetric saddle point problems, Calcolo (to appear).
[8] D.J. Silvester, H.C. Elman, A. Ramage, IFISS: incompressible flow iterative solution software, http://www.manchester.ac.uk/ifiss.